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On Spectral Mapping Theorems

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Let a bounded domain G in $\mathbb{C}^n$ be either strictly pseudoconvex with C^2-boundary $b(G)$ or a polydomain. For each $\boldsymbol{u} = (u_1,..., u_k) \subset H^\infty(G)$, $\mathbf{u} \subset A(G)$, resp. let $\tilde{\sigma}(\mathbf{u})$ be a compact subset of $\mathbb{C}^n$. If so defined $\tilde{\sigma}$ is a subspectrum [i.e., satisfies the projection property and $\tilde{\sigma}(\mathbf{u})$ is a non-void subset of the "usual" joint spectrum of $\mathbf{u}$], then it is shown that $\mathbf{u}(\tilde{\sigma}(\mathbf{z}) \cap G) \subset \tilde{\sigma}(\mathbf{u})$. Moreover, if $\mathbf{u}$ is continuously extendable to each point of $\tilde{\sigma}(\mathbf{z}) \cap b(G)$, then $\mathbf{u}(\tilde{\sigma}(\mathbf{z})) = \tilde{\sigma}(\mathbf{u})$. This provides spectral mapping theorems for $H^\infty(G)$ [resp. $A(G)$]-functional calculi. The extended spectrum of a representation, introduced by C. Foiaş and W. Mlak [*Stud. Math.* **64** (1979), 263–271], is also discussed.

1. INTRODUCTION

The main subject of this work is a spectral mapping theorem for $H^\infty(G)$-functional calculus. It was studied for G as the unit disc and for a completely non-unitary contraction T by Foiaş and Mlak in [4]. Trying to extend their results for G as a polydisc and for the Taylor spectrum of (dilatable) n-tuples of contractions, we realized that the ideas of [4] can be generalized for other types of joint spectra and for a wide class of domains G in $\mathbb{C}^n$. We shall obtain our main results using certain properties of the maximal ideal space of $H^\infty(G)$ and the idea of "functional representation" of spectrum. It is worth remarking that this important idea appeared already in [4] (in a special case).

2. FUNCTIONAL REPRESENTATION OF SUBSPECTRUM

It will be convenient to use the axiomatic approach to joint spectra introduced in [10].

2.1. DEFINITION. Let A be a complex unital Banach algebra. Let $c(A)$ stand for the set of all finite families $\mathbf{x}$ of pairwise commuting elements of A. If B is a fixed commutative, unital Banach subalgebra of A, let $\mathfrak{M}(B)$ denote

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its maximal ideal space equipped with the Gelfand topology. For $\mathbf{x} = (x_\alpha)_{\alpha \in a} \in c(B)$ let $\hat{\mathbf{x}}$: $\mathfrak{M}(B) \to \mathbb{C}^a$ denote its Gelfand transform: $\hat{\mathbf{x}}(h) = (h(x_\alpha))_{\alpha \in a}$. Let $\sigma_B(x) = \hat{\mathbf{x}}(\mathfrak{M}(B))$ be its joint spectrum with respect to B.

2.2. Definition. A mapping $c(A) \ni \mathbf{x} \to \tilde{\sigma}(\mathbf{x})$, where $\tilde{\sigma}(\mathbf{x})$ is a compact subset of $\mathbb{C}^a$ ($=\mathbb{C}^n$ for $\mathbf{x} = (x_\alpha)_{\alpha \in a}$ and $a = \{a_1, \dots, a_n\}$), is called a spectral system on A provided $\emptyset \neq \tilde{\sigma}(x) \subset \sigma(x)$ for all $x \in A$. Here $\sigma(x)$ denotes the spectrum of x as an element of A. Moreover, $\tilde{\sigma}$ is said to possess the projection property, if for each $\mathbf{x} = (x_\alpha)_{\alpha \in a} \in c(A)$ and $\emptyset \neq b \subset a$ we have $\tilde{\sigma}(P(\mathbf{x})) = P(\tilde{\sigma}(\mathbf{x}))$, where P: $\mathbb{C}^a \to \mathbb{C}^b$ is the natural projection.

Remark. In [7] it was shown how to extend $\tilde{\sigma}$ having the projection property from the set $c(A)$ [denoted there by $\mathfrak{c}_0(A)$] to the set of all commutative families in A [denoted in [7, 10] by $\mathfrak{c}(A)$]. All the results of our work hold true for infinite families of elements and mappings, but we shall consider only finite ones.

A class of spectral systems which will be of interest here is described in the following theorem, essentially contained in [7, 10]:

2.4. Theorem. *If $\tilde{\sigma}$ is a spectral system on A which has the projection property, then the following conditions are equivalent*:

(i) $\tilde{\sigma}(P(\mathbf{x})) = P(\tilde{\sigma}(\mathbf{x}))$ *for each polynomial* P: $\mathbb{C}^n \to \mathbb{C}$ *and* $\mathbf{x} = (x_1, \dots, x_n) \in c(A)$.

(ii) *For any maximal commutative subalgebra B of A there exists a compact subset $\Delta = \Delta(\tilde{\sigma}, B)$ of $\mathfrak{M}(B)$ such that $\tilde{\sigma}(\mathbf{x}) = \hat{\mathbf{x}}(\Delta)$ for all $x \in c(B)$.*

(iii) *For any $\mathbf{x} = (x_1, x_2, x_3) \in c(A)$ we have $\tilde{\sigma}(\mathbf{x}) \subset \sigma_B(\mathbf{x})$ [see (2.1)], where B is the unital Banach algebra generated by $\{x_1, x_2, x_3\}$.*

[That (i) implies (ii) follows from the proofs of Theorems 5.1 and 5.3 in [10]; (ii) obviously implies (iii). Finally, (i) is obtained from (iii) in Theorem 3.3 of [7].]

2.5. Definition. We shall call a spectral system $\tilde{\sigma}$ a subspectrum provided that it has the projection property and satisfies the condition (i) of the above theorem. A closed subset Δ of $\mathfrak{M}(B)$ will be said to represent a subspectrum $\tilde{\sigma}$ (with respect to B), if $\tilde{\sigma}(\mathbf{x}) = \hat{\mathbf{x}}(\Delta)$ for all $\mathbf{x} \in c(B)$. The last condition may be called the generalised Gelfand formula for $\tilde{\sigma}$.

Here and in the following remarks B stands for a commutative unital Banach subalgebra of A and $\tilde{\sigma}$ for a subspectrum on A.

Remarks. (1) There exists a unique closed subset $\Delta = \Delta(\tilde{\sigma}, B)$ of $\mathfrak{M}(B)$

such that $\tilde{\sigma}(\mathbf{x}) = \hat{\mathbf{x}}(\Delta)$ for each $\mathbf{x} \in c(B)$. In [10, Theorem 2.3] this set Δ [denoted there by $\tilde{\sigma}(B)$] is described as follows:

$$\Delta(\tilde{\sigma}, B) = \{h: B \to \mathbb{C}; (h(x_1),..., h(x_n)) \in \tilde{\sigma}(\mathbf{x}) \quad \text{if } \mathbf{x} = (x_1,..., x_n) \in c(B)\}.$$

(2) Let $\mathbf{x} \in c(B)$ and let $\mathbf{f} = (f_j)_{j=1,...,m}$ be a family of functions holomorphic in the neighborhood of $\sigma_B(\mathbf{x})$. The Shilov–Arens–Calderón functional calculus gives us an m-tuple $(f_j(\mathbf{x}))_{j \leqslant m} = \mathbf{f}(\mathbf{x}) \in c(B)$ such that $(\mathbf{f}(\mathbf{x}))\hat{} = \mathbf{f} \circ \hat{\mathbf{x}}$. Now, the Gelfand formula for $\tilde{\sigma}$ yields $\tilde{\sigma}(\mathbf{f}(\mathbf{x})) = \mathbf{f}(\tilde{\sigma}(\mathbf{x}))$, the spectral mapping theorem for such an $\mathbf{f}$. In the case of the reacher (as shown in [1]) functional calculus of Taylor [9] this is not so trivial. However, if X is a Hilbert space, $B \subset \mathscr{L}(X)$ and the f_j are holomomorphic near $\sigma_T(\mathbf{x})$, then for any $h \in \Delta(\sigma_T, B)$ we have $(f_j(\mathbf{x}))\hat{}(h) = f_j(\hat{\mathbf{x}}(h))$. It was proved by Vasilescu in [*Rev. Roum. Math. Pures Appl.* **23** (1978), 1587–1605], where a new approach to Taylor's functional calculus was presented. Therefore, if only $\Delta(\tilde{\sigma}, B) \subset (\sigma_T, B)$, we have a similar spectral mapping theorem for $\tilde{\sigma}$ and for families $\mathbf{f}$ of functions holomorphic near $\sigma_T(\mathbf{x})$.

(3) The investigation of $\tilde{\sigma}$ may often be reduced to looking for properties of $\Delta(\tilde{\sigma}, B)$, although this set is difficult to find in general.

(4) For the (bi) commutant spectrum the projection property fails. Except for these two, most of the known spectral systems are subspectra. We list some of them below (see [10] for further details):

—The left spectrum σ_l defined with respect to a fixed unital Banach algebra A; similarly, the right spectrum σ_r and the spectrum $\sigma(\mathbf{x}) = \sigma_l(\mathbf{x}) \cup \sigma_r(\mathbf{x})$.

—If X is a complex Banach space and $A = \mathscr{L}(X)$ is the algebra of bounded endomorphisms of X, then we have σ_π—the approximate point spectrum, σ_{d}—the defect spectrum, and σ_{T}—the Taylor spectrum (see [8]).

Let $h: A \to A_1$ be a homeomorphism of unital Banach algebras. Having a spectral system $\tilde{\sigma}$ on A_1, define $(h^*\tilde{\sigma})(\mathbf{x}) = \tilde{\sigma}(h\mathbf{x})$, where $\mathbf{x} = (x_\alpha)_{\alpha \in a} \in c(A)$ and $h\mathbf{x} = (hx_\alpha)_{\alpha \in a}$. As shown in [10], if $\tilde{\sigma}$ is a subspectrum, then $h^*\tilde{\sigma}$ has the same property. It is easy to see that $\Delta(h^*\tilde{\sigma}, A) = \{s \circ h; s \in \Delta(\tilde{\sigma}, A_1)\}$. In this way we can obtain certain subspectra, for example, the (left) essential spectrum and the subspectrum investigated in Section 4.

3. Theorems for $H^\infty(G)$-Functional Calculus

In this section we shall deal with algebras of functions in several complex variables. The results concerning their maximal ideal spaces are scattered in the literature, so we quote some of them for the convenience of the reader. Let us fix the notation. Let G denote an open, bounded subset of $\mathbb{C}^n$. $H^\infty(G)$

will stand for the Banach algebra of bounded holomorphic functions on G (with the sup-norm), and $A(G)$ for its subalgebra of functions having continuous extensions onto $\bar{G}$, the closure of G. Since the coordinate functions z_j $(j=1,\dots,n)$ belong to these (commutative, unital Banach) algebras, we have a natural projection Π from $\mathfrak{M}(G):=\mathfrak{M}(H^\infty(G))$ into $\mathbb{C}^n$ given by $\Pi(h)=(h(z_1),\dots,h(z_n))$ and, similarly, $\bar{\Pi}\colon \mathfrak{M}(A(G))\to\mathbb{C}^n$. The fiber of $\mathfrak{M}(G)$ over $t\in\mathbb{C}^n$, denoted $\mathfrak{M}_t(G)$, is defined as $\Pi^{-1}\{t\}$ (pre-image). We have the natural embeddings: G in $\mathfrak{M}(G)$ and $\bar{G}$ in $\mathfrak{M}(A(G))$ defined by $t\to\delta_t$, where $\delta_t(u)=u(t)$. Here the question arises whether $\mathfrak{M}_t(G)=\{\delta_t\}$ for $t\in G$. Of course, this is so if for any $u\in H^\infty(G)$ the following condition holds true:

$(+)$ there exist $g_1,\dots,g_n$ in $H^\infty(G)$ such that

$$u(\mathbf{z})-u(t)=\textstyle\sum_{j\leqslant n}(z_j-t_j)\,g_j(\mathbf{z}) \text{ for all } \mathbf{z}\in G.$$

Note that $(+)$ becomes trivial only for G of the form $G_1\times\cdots\times G_n$, where G_j's are bounded plane domains. Such a set G will be called a bounded polydomain. On the other hand, Henkin has proved in [6] that if $G\subset\mathbb{C}^n$ is a strictly pseudoconvex domain having C^3-smooth boundary $b(G)$, then for any $u\in A(G)$ and $t\in G$ the condition $(+)$ holds true with $g_j\in A(G)$ $(j=1,\dots,n)$. (See [12] for definitions.) The construction carried out in [6] is applicable for $u\in H^\infty(G)$ and it is easy to see that in this case we shall obtain g_j's from $H^\infty(G)$ satisfying $(+)$, even if G has only the C^2-boundary. For example, if G is convex and $t=0$, we can write

$$u(z)-u(0)=\int_0^1\frac{du(sz)}{ds}\,ds=\sum_{k=1}^n z_k\,g_k(z)\qquad\text{with}\quad g_k(z)=\int_0^1\frac{\partial u(sz)}{\partial z_k}\,ds.$$

The last integrand is uniformly bounded on G by $M(1-s)^{-1/2}$ for $b(G)$ is smooth (Cauchy Inequality is also in use); so $g_k\in H^\infty(G)$ for $k<n$. This was kindly communicated to me by Dr. Jakóbczak, who also noted that the methods of [11] (related to the $\bar{\partial}$-problem) enable us to obtain the same for G as a Weyl (polynomial) polyhedron. Now, using the notation introduced above, we may formulate:

3.1. THEOREM. *Let G be a bounded polydomain (or a strictly pseudoconvex domain in $\mathbb{C}^n$ with C^2-boundary). Then $\mathfrak{M}_t(G)=\{\delta_t\}$ for each $t\in G$. The restriction $\Pi|_{\Pi^{-1}G}$ is injective.*

To describe the fiber $\mathfrak{M}_t(G)$ for $t\notin G$ we shall need more information. The first piece of information is that $\mathfrak{M}(A(G))=\bar{G}$ (i.e., $=\{\delta_t;\, t\in\bar{G}\}$), for strictly pseudoconvex (s.psc.) domain G with smooth boundary. It was proved in [13]. Since the restriction of a homomorphism belonging to $\mathfrak{M}_t(G)$ is an element of $\mathfrak{M}(A(G))$, one easily deduces that $\mathfrak{M}_t(G)=\varnothing$ for $t\notin\bar{G}$. Finally, let $t \in b(G)$ $(=\bar{G} \backslash G)$. For $w \in H^\infty(G)$ the set $\hat{w}(\mathfrak{M}_t(G))$ may be quite large, because it obviously contains $\mathrm{cl}(w; t)$—the cluster set of w at t (defined as $\{y \in \mathbb{C}; w(t_n) \to y$ for a certain sequence $\{t_n\}$ converging to t in $G\}$). Let us define w to be continuously extendable to a point $t \in b(G)$, if $\mathrm{cl}(w; t)$ has only one element (say, y). Then, putting $w(t)$ for y, one obtains a continuous on $G \cup \{t\}$ function, which extends w. The existence of a peak function $h \in A(G)$ at t implies in this case that $w(\mathfrak{M}_t(G)) = \{w(t)\}$. If G is s.psc., with C^2-boundary, the existence of such h [i.e., $h(1) = 1$, $|h(z)| < 1$ for $\mathbf{z} \in \bar{G} \backslash \{t\}$] is proved in [13] (for example). Now, if $w(t) = 0$, then $h^n w$ (uniformly on G) tends vers 0, and $\varphi(h^n w) = (\varphi h)^n (\varphi w) \to 0$ for any $\varphi \in \mathfrak{M}_t(G)$. But $\varphi h = 1$; so it must be $\varphi w = 0$ and we are done. Moreover, Lemma 7 of [14] (proved there for a ball, but valid also for a s.psc. G) says that if for $w \in H^\infty(G)$, $0 \notin \mathrm{cl}(w; t)$, then one can find $u \in H^\infty(G)$ such that $\mathrm{cl}(uw; t) = \{1\}$. Therefore the following theorem, proved by Gamelin [2, Theorem 7.5] for G as a bounded polydomain can be stated also for a s.psc. G with C^2-boundary (as suggested in [14]).

3.2. THEOREM. *If G is as in Theorem* 3.1, *$t \in \mathbb{C}^n$ and $w \in H^\infty(G)$, then $\hat{w}(\mathfrak{M}_t(G)) = \mathrm{cl}(w; t)$, in particular, $=\varnothing$ for $t \notin \bar{G}$, $=\{w(t)\}$, if w is continuously extendable to $t \in \bar{G}$.*

We shall apply this description of $\mathfrak{M}(G)$ in spectral theory of $H^\infty(G)$, denoted by A, for convenience. Let $\tilde{\sigma}$ be a subspectrum on A $(=H^\infty(G))$. The projection $\Pi: \mathfrak{M}(G) \to \bar{G}$ equals $\hat{\mathbf{z}}$, the Gelfand transform of $\mathbf{z} = (z_1, \ldots, z_n)$; so if Δ stands for $\Delta(\tilde{\sigma}, A)$, the representing set for $\tilde{\sigma}$, then $\Pi(\Delta) = \tilde{\sigma}(\mathbf{z})$, by Definition 2.5.

3.3. PROPOSITION. *If G is as in Theorem* 3.1, *then $\Delta \cap \Pi^{-1}G = \Pi^{-1}(\tilde{\sigma}(\mathbf{z}) \cap G)$ and $\mathbf{u}(\tilde{\sigma}(\mathbf{z}) \cap G) \subset \tilde{\sigma}(\mathbf{u})$ for any $\mathbf{u} = (u_1, \ldots, u_k) \subset H^\infty(G)$. Moreover, if $\bar{\tilde{\sigma}}$ is a subspectrum on $A(G)$, then $\bar{\tilde{\sigma}}(u) = \mathbf{u}(\bar{\tilde{\sigma}}(\mathbf{z}))$ for each $\mathbf{u} \subset A(G)$.*

Proof. Π is injective on $\Pi^{-1}G$, by Theorem 3.2, so $\Pi(\Delta \cap \Pi^{-1}G) = \Pi(\Delta) \cap \Pi(\Pi^{-1}G) = \tilde{\sigma}(\mathbf{z}) \cap G = \Pi(\Pi^{-1}(\tilde{\sigma}(\mathbf{z}) \cap G))$, hence the first assertion. Therefore each $t \in \tilde{\sigma}(\mathbf{z}) \cap G$ is of the form $\Pi(h)$, where $h \in \Delta$ and $h = \delta_t$. Now, $\mathbf{u}(t) = \hat{\mathbf{u}}(h) \in \hat{\mathbf{u}}(\Delta) = \tilde{\sigma}(\mathbf{u})$, by 2.4. The last assertion follows from the equalities: $\mathfrak{M}(A(G)) = \bar{G}$, $\Delta(\bar{\tilde{\sigma}}, A(G)) = \{\delta_t; t \in \bar{\tilde{\sigma}}(z)\}$. Hence $\bar{\tilde{\sigma}}(u) = \hat{u}(\Delta(\bar{\tilde{\sigma}}, A(G))) = u(\bar{\tilde{\sigma}}(z))$.

The image $u(\tilde{\sigma}(\mathbf{z}))$ makes no sense in general for $u \in H^\infty(G)$ if $\tilde{\sigma}(\mathbf{z}) \cap b(G) \neq \varnothing$, for u is defined on G only. However, we may define $u(\tilde{\sigma}(\mathbf{z}))$ to be the union $\bigcup_{t \in \tilde{\sigma}(z)} \mathrm{cl}(u; t)$. This notation is especially motivated, if u is continuously extendable to each $t \in \tilde{\sigma}(\mathbf{z})$, as the following theorem shows.

3.4. THEOREM. *Let G be a domain in $\mathbb{C}^n$, regular as in Theorem* 3.1. *Let*

$\tilde{\sigma}$ be a subspectrum on $H^\infty(G)$. If $\mathbf{u} = (u_1, \ldots, u_k) \subset H^\infty(G)$ is such that each u_j is continuously extendable to any point $t \in \tilde{\sigma}(\mathbf{z}) \cap b(G)$, then $\tilde{\sigma}(\mathbf{z}) \subset \bar{G}$ and (regarding u_j's to be so extended) we have $\tilde{\sigma}(\mathbf{u}) = \mathbf{u}(\tilde{\sigma}(\mathbf{z}))$. (Precisely, the last set is defined as $\{(u_1(t), \ldots, u_k(t));\ t \in \tilde{\sigma}(\mathbf{z})\}$ if $\mathrm{cl}(u_j; t) = \{u_j(t)\}$.)

Proof. Let $\Delta = \Delta(\tilde{\sigma}, H^\infty(G))$. Since $\mathfrak{M}_t(G) = \varnothing$ for $t \notin \bar{G}$ (by Theorem 3.2) and $\mathfrak{M}(G) = \bigcup_{t \in \mathbb{C}^n} \mathfrak{M}_t(G)$, we have $\tilde{\sigma}(\mathbf{z}) = \Pi(\Delta) \subset \Pi(\mathfrak{M}(G)) \subset \bar{G}$. Of course, $\Delta \cap \mathfrak{M}_t(G) \neq \varnothing$ if and only if $t \in \tilde{\sigma}(\mathbf{z})$; so $\Delta = \bigcup_{t \in \tilde{\sigma}(\mathbf{z})} (\Delta \cap \mathfrak{M}_t(G))$. If $t \in \tilde{\sigma}(\mathbf{z})$, then $\hat{u}_j(\mathfrak{M}_t(G)) = \{u_j(t)\}$ for $j = 1, \ldots, k$, by Theorem 3.2; so $\hat{\mathbf{u}}(\mathfrak{M}_t(G)) = \{\mathbf{u}(t)\}$. Now $\tilde{\sigma}(\mathbf{u}) = \hat{\mathbf{u}}(\Delta) = \bigcup_{t \in \tilde{\sigma}(\mathbf{z})} \{\mathbf{u}(t)\} = \mathbf{u}(\tilde{\sigma}(\mathbf{z}))$.

Remark. This theorem may be interpreted as a spectral mapping theorem for an $H^\infty(G)$-functional calculus. Indeed, a subspectrum $\tilde{\sigma}$ often arises as follows: Let $H^\infty(G) \ni u \to u(T_1, \ldots, T_n) \in \mathscr{L}(X)$ be a certain functional calculus in operators $T_1, \ldots, T_n$ on a complex Banach space X. It means, that $u \to u(T_1, \ldots, T_n)$ is a representation of the algebra $H^\infty(G)$ in X "preserving the unit" ("unital") and such that $z_j(T_1, \ldots, T_n) = T_j$ $(j \leqslant n)$ [For example, if $(T_1, \ldots, T_n)$ is an n-tuple of contractions in Hilbert space, satisfying the von Neumann Inequality and G is the unit polydisc in $\mathbb{C}^n$, then such a calculus was constructed in [3].]

Now, if $\bar{\sigma}$ is a subspectrum on $\mathscr{L}(X)$, then a subspectrum $\tilde{\sigma}$ on $H^\infty(G)$ can be defined by $\tilde{\sigma}(\mathbf{u}) = \bar{\sigma}(\mathbf{u}(T_1, \ldots, T_n))$ for $\mathbf{u} = (u_1, \ldots, u_k) \subset H^\infty(G)$ (cf. Remark 4 in Section 2.) In the last theorem, the formula $\tilde{\sigma}(\mathbf{u}) = \mathbf{u}(\tilde{\sigma}(\mathbf{z}))$ becomes $\bar{\sigma}(\mathbf{u}(T_1, \ldots, T_n)) = \mathbf{u}(\bar{\sigma}(T_1, \ldots, T_n))$, i.e., the spectral mapping for $\mathbf{u} \subset H^\infty(G)$ and continuously extendable to each point of $\bar{\sigma}(T_1, \ldots, T_n) \cap b(G)$. Similarly, in this notion, Proposition 3.3 reads $\mathbf{u}(\bar{\sigma}(T_1, \ldots, T_n) \cap G) \subset \bar{\sigma}(\mathbf{u}(T_1, \ldots, T_n))$ for each $\mathbf{u} \subset H^\infty(G)$ (G as in Theorem 3.1).

Now we shall give an example concerning the assumption on $\mathbf{u}$ in Theorem 3.4 [u_j's were regarded to be continuously extendable to each point of $\tilde{\sigma}(\mathbf{z}) \cap b(G)$ and, consequently, of $\tilde{\sigma}(\mathbf{z})$). Let us consider the case when $k = 1$ and G is the unit polydisc: $D \times D$ in $\mathbb{C}^2$. Let $\partial G = \partial D \times \partial D$ (distinguished boundary). One can ask whether $u(\tilde{\sigma}(\mathbf{z})) = \tilde{\sigma}(u)$ for $u \in H^\infty(G)$ which is continuously extendable to each point of $\tilde{\sigma}(z) \cap \partial G$. The answer is "no."

Example. In [4] it was shown that there exist $v \in H^\infty(D)$ and a completely non-unitary contraction $S \in \mathscr{L}(H)$ such that $\sigma(S) = \{1\}$, $\mathrm{cl}\,(v; 1) = \bar{D}$, but $v(S) = 0$. If $u \in H^\infty(D \times D)$, let $u_0(z) = u(z, 0)$ and let $u(S) = u_0(S)$—in the sense of the Nagy–Foiaş functional calculus $[u_0 \in H^\infty(D)]$. If R is the resolvent algebra for the representation $H^\infty(D \times D) \ni u \to u(S)$ (see Section 4) and if $\tilde{\sigma}(u_1, \ldots, u_k) = \sigma_R(u_1(S), \ldots, u_k(S))$, then $\tilde{\sigma}$ is a subspectrum on $H^\infty(D \times D)$ (cf. Section 2).

We have $\tilde{\sigma}(z) = \sigma_R(S, 0) \subset \sigma_R(S) \times \{0\} = \{(1, 0)\}$, so $\tilde{\sigma}(z) \cap \partial(D \times D) = \varnothing$. The function $f(z, w) := v(z)$ trivially satisfies the last assumption, but $f(S) = 0$, so $\tilde{\sigma}(f) = \{0\} \neq \bar{D} = \mathrm{cl}(v; 1) = \mathrm{cl}(f; (1, 0))$.

4. The Extended Spectrum

This work originated from the ideas of [4]. Let us recall them in a slightly more general setting. We shall consider a unital representation $T: A \to \mathscr{L}(X)$ of a commutative Banach algebra A in a Banach space X (over $\mathbb{C}$). The extended spectrum of $T(\cdot)$ was defined as $\sigma_{\mathrm{ext}}(T) := \{h \in \mathfrak{M}(A);\ h(u) \in \sigma(T(u))$ for all $u \in A\}$. [In [4] $T(\cdot)$ was the functional calculus of Nagy and Foiaş, $T(u) := u(T)$, where T is a completely non-unitary contraction and $u \in H^\infty$ $(=H^\infty(D))$.] The resolvent algebra R for $T(\cdot)$ is defined as the Banach algebra generated in $\mathscr{L}(X)$ by $\{(T(u))^{-1};\ u \in A,\ 0 \notin \sigma(T(u))\}$. For $h \in \mathfrak{M}(R)$ let $h \circ T \in \mathfrak{M}(A)$ be such that $\hat{u}(h \circ T) = (T(\mathbf{u}))\hat{}\ (h)$ for $u \in A$ and let $\sigma_0 = \{h \circ T;\ h \in \mathfrak{M}(R)\}$.

One can easily see (cf. (4)) that $T(\mathbf{u}) \in R$ and $\sigma_R(T(\mathbf{u})) = \sigma(T(\mathbf{u}))$ for each $u \in A$. Moreover, σ_0 and $\sigma_{\mathrm{ext}}(T)$ are compact subsets of $\mathfrak{M}(A)$ and $\sigma_0 \subset \sigma_{\mathrm{ext}}(T)$.

Let $\tilde{\sigma}(u_1, \ldots, u_k) = \sigma_R(T(u_1), \ldots, T(u_k))$ for $\mathbf{u} = (u_1, \ldots, u_k) \subset A$. Let $T(\mathbf{u}) = (T(u_1), \ldots, T(u_k))$ for $\mathbf{u}$ as above. We have $\sigma_0 = \Delta(\tilde{\sigma}, A)$, by Remark 1 in Section 2, since $\sigma_R(T(\mathbf{u})) = (T(\mathbf{u}))\hat{}\ \mathfrak{M}(R) = \hat{\mathbf{u}}(\sigma_0)$ for any $\mathbf{u} \subset A$.

4.1. Proposition. $\sigma_0 = \sigma_{\mathrm{ext}}(T)$.

Proof. It suffices to show that $\hat{\mathbf{u}}(\sigma_{\mathrm{ext}}(T)) \subset \sigma_R(T(\mathbf{u}))$ for each $\mathbf{u} = (u_1, \ldots, u_k) \subset A$. By denial: Let us suppose that there exists $h \in \sigma_{\mathrm{ext}}(T)$ such that $\hat{\mathbf{u}}(h) \notin \sigma_R(T(\mathbf{u}))$ for a certain $\mathbf{u} \subset A$. Then there exist $B_1, \ldots, B_k \in R$ satisfying $\sum B_j(T(u_j) - u_j(h) I) = I$ (the identity operator on X). But it is easy to see that R is the norm-closure of $\{T(w)(T(v))^{-1};\ w, v \in A,\ 0 \notin \sigma(T(v))\}$; so we can find $w_j, v_j \in A$ $(j = 1, \ldots, k)$ such that an operator $S := \sum (T(u_j) - \hat{u}_j(h) I)\, T(w_j)(T(v_j))^{-1}$ is invertible (choose them to have $\|S - I\| < 1$). Now, multiplying S by $\prod_{j=1}^k T(v_j)$ (invertible), we shall obtain that there exists $w \in A$ such that $\hat{w}(h) = 0$, but $0 \notin \sigma(T(w))$ [because $T(\cdot)$ is a unital representation]. Hence a contradiction: $h \notin \sigma_{\mathrm{ext}}(T)$.

5. Several Remarks

Now we shall consider a relationship between $\Delta(\tilde{\sigma}, A)$ and ∂—the Šilov boundary of A in the case $A = H^\infty(G)$. If E is a subset of G, $u \in H^\infty(G)$, let $|u|_E = \sup_{z \in E} |u(z)|$. E is said to be dominating (in G) if $|u|_E = |u|_G$ for all $u \in H^\infty(G)$. Let G be a domain in $\mathbb{C}^n$, regular as in Theorem 3.1.

—If $\tilde{\sigma}(\mathbf{z}) \cap G$ is dominating, then $\partial \subset \Delta(\tilde{\sigma}, A)$ (apply Proposition 3.3)

—As the example of bilateral shift N shows it may happen that $\partial \subset \Delta(\tilde{\sigma}, A)$, but $\tilde{\sigma}(\mathbf{z}) \cap G = \varnothing$ [put $G = D$, $T(u) = u(N)$, for $u \in H^\infty(G)$, $\tilde{\sigma}$ as in Section 4].

—On the other hand, if $\partial \subset \Delta(\tilde{\sigma}, A)$, then for each $u \in A$ $(=H^\infty(G))$ $|u|_G = |u|_{\tilde{\sigma}(z)}$ provided that u is continuously extendable to each point of $\tilde{\sigma}(z) \cap b(G)$ (by Theorem 3.4).

—The converse is not true, as the following example shows:

As shown in [5], there exist a contraction S and an inner function w satisfying $\sigma(S) = \partial D$ (the unit circle), but $w(S) = 0$ (i.e., $S \in \mathscr{C}_0$). Let $T(u) = u(S)$ for $u \in H^\infty$. If $\tilde{\sigma}$ is as in Section 4, then $\Delta(\tilde{\sigma}, H^\infty)$ is disjoint with ∂ the Šilov boundary of H^∞, while $\tilde{\sigma}(z) = \partial D$. Indeed, $\hat{w}(\Delta(\tilde{\sigma}, H^\infty)) = \sigma(w(S)) = \{0\}$, $w = 1$, on ∂ (by J. Newmann's Theorem).

Similarly one may prove that $\partial \cap \Delta(\tilde{\sigma}, H^\infty) = 0$ if and only if there exist inner functions $u_1, \dots, u_k$ such that $\tilde{\sigma}(u_1, \dots, u_k) \cap (\partial D)^k = \varnothing$.

At the end of this work we present an application of Proposition 4.1. Operators $S_j \in \mathscr{L}(H_j)$ $(j = 1, 2)$ are said to be quasi-similar if there exist operators $Y_{ij}\colon H_j \to H_i$ which are quasi-invertible (i.e., injective, with dense ranges) such that $S_i Y_{ij} = Y_{ij} S_j$ for $i, j = 1, 2$ and $i \neq j$. By the result of S. Clary, the spectra of quasi-similar hyponormal operators are equal [*Proc. Amer. Math. Soc.* **53** (1975), 88].

Let $T_i(\cdot)$ be a unital representation of a commutative Banach algebra A in a Hilbert space H_i for $i = 1, 2$. As an immediate consequence of Proposition 4.1 and Clary's Theorem we have:

5.1. Proposition. *If for each $u \in A$ operators $T_1(u)$ and $T_2(u)$ are quasi-similar and hyponormal, then $\sigma_{\text{ext}}(T_1) = \sigma_{\text{ext}}(T_2)$. Consequently, if R_i denotes the corresponding resolvent algebra for $T_i(\cdot)$ (see Section 4), then for each $w = (w_1, \dots, w_k) \subset A$ the joint spectra $\sigma_{R_i}(T_i(w_1), \dots, T_i(w_k))$ $(i = 1, 2)$ are equal.*

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